Mass Center - Rigid Body

Rigid Body

• A rigid body can be thought of as a “continuum,” a solid collection of an infinite number of particles, each with infinitesimal mass, rigidly “glued” together.

• To find the first mass moment of a rigid body, we must sum an infinite number of first moments of particles!

• But you have encountered infinite sums before in calculus.

FMM: Rigid Body

In the limit as the number of particles becomes infinite and the largest particle becomes infinitesimal, the first mass moment becomes an integral.

\[ \lim_{N \to \infty} \left( \sum_{n=1}^{N} m_n \bar{r}_n \right) = \int \bar{r} \, dm \]

The \( m \) below the integral denotes the domain over which the integral is evaluated, which is the entire mass of the body.

Mass Center: Rigid Body

\[ \bar{r}_G = \sum_{n=1}^{N} m_n \bar{r}_n \rightarrow \int \bar{r} \, dm = \frac{1}{m} \int \bar{r} \, dm \]
Mass Center Properties

• The position of the mass center (PMC) is the average position of all the mass.
• The PMC is the FMM of the system divided by the total mass.
• At the mass center, the FMM is zero.
• If planes of symmetry exist, the mass center is in them.

Centroid: Rigid Body

If a rigid body has "uniform" (constant throughout the body) mass density, then its mass center and centroid coincide.

\[ dm \rightarrow \rho dV \]

\[
\overline{r}_G = \frac{1}{V} \int_V \overline{r} \, dV = \frac{1}{\rho V} \int_V \rho \overline{r} \, dV = \frac{1}{V} \int_V \overline{r} \, dV = \overline{r}_C
\]

Composite Bodies - 1

• Because the integral of a sum equals the sum of the integrals, we can separate integrals into pieces, evaluate the pieces, and then add the pieces back together.
• Sometimes we can evaluate the pieces from simple formulas in tables.
Composite Bodies - 2

• Identify the **simple shapes** in the body.

• Write the FMM of the entire body as the **sum** of the FMM’s of the simple shapes.

• Replace the FMM of each simple shape with the product of its mass and the **position of its mass center**.

• Add the FMM of the simple shapes and divide by the **total mass**.

 Locate the mass center for a uniformly dense mallet modeled as the stick-cube composite body shown.

From symmetry, \( y_G = z_G = 0 \)

\[
x_G = \frac{1}{m} \int x \, dm = \frac{\int x \, rd_{rod} + \int x \, dcube}{m_{rod} + m_{cube}} = \frac{m_{rod} \left( \frac{L}{2} \right) + m_{cube} \left( L + \frac{s}{2} \right)}{m_{rod} + m_{cube}}
\]

\[
m_1 = m_{rod} = \rho V_{rod} = \rho (\pi R^2 L)
\]

\[
m_2 = m_{cube} = \rho V_{cube} = \rho s^3
\]

\[
x_G = \frac{\sum_{n=1}^{2} m_n x_Gn}{m_1 + m_2}
\]

Where is the mass center of the uniformly thick angle iron? (In this case, thickness is the dimension normal to the plane of the figure.)

\[
\vec{r}_G = \frac{1}{m} \int \vec{r} \, dm = \frac{1}{m} \left( \sum_{n=1}^{N} m_n \vec{r}_{Gn} \right)
\]

\[
x_G \hat{i} + y_G \hat{j} = \frac{1}{A} \left( \sum_{n=1}^{N} A_n (x_n \hat{i} + y_n \hat{j}) \right)
\]

\[
m = \rho V = \rho At
\]

\[
m_n = \rho V_n = \rho A_n t
\]

\[
x_G = \bar{x} = \frac{1}{A} \left( \sum_{n=1}^{N} A_n \bar{x}_n \right)
\]

\[
y_G = \bar{y} = \frac{1}{A} \left( \sum_{n=1}^{N} A_n \bar{y}_n \right)
\]
For a very thin angle iron, \( x_G \to 250 \text{ mm} \) 
\( y_G \to 350 \text{ mm} \)

The mass is
\[ m = m_{\text{cube}} - m_{\text{hole}} = \int_{\text{cube}} dm - \int_{\text{hole}} dm \]

The \( z \) component of the FMM is
\[ \int_{m} z \, dm = \int_{\text{cube}} z \, dm - \int_{\text{hole}} z \, dm = m_{\text{cube}} z_{G_{\text{cube}}} - m_{\text{hole}} z_{G_{\text{hole}}} \]

Then the \( z \) coordinate of the mass center’s position is
\[ z_G = \frac{m_{\text{cube}} \left( \frac{a}{2} \right) - m_{\text{hole}} \left( \frac{a}{2} + \frac{h}{2} \right)}{m_{\text{cube}} - m_{\text{hole}}} = \frac{V_{\text{cube}} \left( \frac{a}{2} \right) - V_{\text{hole}} \left( \frac{a}{2} + \frac{h}{2} \right)}{V_{\text{cube}} - V_{\text{hole}}} \quad \text{(Uniform mass density cancels.)} \]

\[ z_G = -\frac{\pi r^2 h \left( \frac{a}{2} + \frac{h}{2} \right)}{a^2 - \pi r^2 h} \]
Mass Center - Rigid Body

2 When we look closely, we find that all physical materials are discontinuous, which means that they have voids (aka spaces) occupied by other materials such as air, water, etc. But on the ordinary length scales sensed by the human eye, material that we call “solid” appears to be continuous. For most engineering courses, you will find that all materials, whether solid, liquid or gas, are analyzed as if they were continuous.

3 The process of integration is something that you have encountered in calculus. As we divide the body into more and more pieces, each piece becomes smaller, until finally each is an infinitesimal mass, $dm$.

4 For both a system of discrete particles and a rigid body, the position of the mass center is the first mass moment of the system divided by the total mass of the system. But the method of calculation depends on the type of system. At the left is the equation derived in previous notes for the position of the mass center of a system of $N$ particles. In the transition from a system of discrete particles to a continuous rigid body, the finite sums in the numerator and denominator become integrals.

5 The properties of the mass center of a rigid body are exactly the same as the properties of the mass center of a system of discrete particles. But the calculations involve integrals, rather than finite sums.
Mass density is mass per unit volume. “Uniform” mass density is the same at every point in the body. Because the mass density does not vary from point to point, it is the same value for every infinitesimal volume $dV$ and can be taken outside the integral. Because it appears in both the numerator and the denominator, the mass density cancels from the equation for the position of the mass center. Then the location of the mass center is the same as the geometric center, which is also known as the centroid.

Each integration produces one quadratic term and two linear terms. In the end, the volume $abc$ cancels.

The first statement here allows us to separate a complicated shape into pieces, compute the FMM for each piece and then sum the results. For a piece that is a simple shape, we can locate the mass center and compute the FMM by multiplying the mass by the location of the mass center. This avoids the need to integrate over the mass of the simple piece. If we can separate the entire body into only simple shapes, we can avoid integration entirely.

The process described here is somewhere in between finding the mass center of a system of discrete particles and finding the mass center of a rigid body by integration. In the former case, we divide one finite sum by another. In the latter case, we integrate over the entire mass of the body. In the process described here, we divide one finite sum by another, but the terms in the sums come from integrals that someone has previously
computed. The masses and mass center positions needed in the third step on this slide must originally have been found by integration or by symmetry. The process is illustrated on the following slide.

10 Following the steps on the previous slide, we break the mallet into two pieces, a rod and a cube. Then the numerator in the first equation becomes the sum of the first moments of the rod and the cube about \( O \). The locations of the mass centers of the rod and the cube can both be found by symmetry, and the masses come from integration or a table. Given both masses and mass center locations, we need only evaluate the quotient of two finite sums. Integration is entirely avoided.

11 This is the final result from the previous slide. It would be useful if we were required to design a simple mallet.

12 To begin, we add a coordinate system. The body has both uniform mass density and uniform thickness. Both the density and the thickness cancel, leaving calculations with areas. A horizontal bar above a coordinate is used here to denote a property of the centroid. For each area in the final two equations, we need to find the \( x \) and \( y \) coordinates of the area’s centroid.
13 The first two equations are results from the previous slide. We could use a horizontal line to cut the body into two rectangles. Or we could use two cuts to make three rectangles. Or we could use a vertical cut to make two rectangles. Using the vertical cut gives the results shown.

If the both legs were very thin, the mass center’s coordinates would be halfway along each leg. The given widths of 80 mm and 100 mm bring the mass center significantly closer to the origin.

14 We have seen that dividing a body into pieces and summing known results for the parts can avoid integration. Here we see that we can also subtract integrals over a hole. There is no material in the hole, so we should not integrate over that region. But rather than perform integration over this complicated shape, we can again use what we know about two simple shapes (a cube and a cylinder) to avoid integration. The new feature here is that one of the shapes is a region that contains no mass. It follows that we should not integrate over that region. But that is equivalent to integrating over the cube and subtracting integration over the hole. The result is given on the next slide.

15 By subtracting the mass and FMM of the hole, we reduce the problem to a simple calculation that does not require integration. Instead, we have finite sums with terms that come from integrals that we already know.