Second Moment of Area

ME 202

Misnomer

• Most people do not use the name “second areal moment” (SAM) or “second moment of area.”

• Instead, most people use “moment of inertia,” even though a SAM is purely geometric and does not involve mass, which is the physical property associated with inertia.

• We have already studied the real moment of inertia.

Generalized Moment

• The second moment of anything about a point $O$ is the product of two things:

  • the square of the distance from $O$ to the anything and

  • the anything itself.

Definitions

Second moment of area $A$ about the $x$ axis:

$$I_x = \int_A (y^2 + z^2) \, dA$$

Second moment of area $A$ about the $y$ axis:

$$I_y = \int_A (x^2 + z^2) \, dA$$

Second moment of area $A$ about the $z$ axis:

$$I_z = \int_A (x^2 + y^2) \, dA$$
Restriction
In this class, all the areas we will consider are in the $xy$ plane. This simplifies two of the previous definitions to

\[ I_x = \int_A y^2 \, dA \]
\[ I_y = \int_A x^2 \, dA \]

Polar Second Moment
• The second moment $I_z$ is sometimes called the **polar** second moment or **polar** moment of inertia, which is often denoted by $J_O$.

• Given the restriction on the previous slide,

\[ I_z = J_O = I_x + I_y \]

Transfer Theorem - 1
• We can “transfer” the second moment (moment of inertia) of an area from one **axis** to another, provided that the two axes are **parallel**.

• In other words, if we know the second moment about one **axis**, we can compute it about another **parallel axis**.

Transfer Theorem - 2
If the second moment of an area $A$ about an axis $x'$ through the centroid is $I_{Cx'}$, and the distance from the $x'$ axis to the (parallel) axis $x$ is $d$, then the second moment of the area about the $x$ axis is

\[ I_x = I_{Cx'} + \frac{Ad^2}{\text{transfer term}} \]
Transfer Theorem - 3

- The second moment to which the transfer term is added is always the one for an axis through the centroid.
- The second moment about an axis through the centroid is the smallest second moment about any other parallel axis.

Transfer Theorem - 4

We can transfer from any axis to a parallel axis through the centroid by subtracting the transfer term.

\[ I_{cx'} = I_x - Ad^2 \]

Radius of Gyration

- By definition, the radii of gyration of an area \( A \) about the \( x \) and \( y \) axes are

\[ k_x = \sqrt{\frac{I_x}{A}} \quad k_y = \sqrt{\frac{I_y}{A}} \]

- Given the area and the radii of gyration,

\[ I_x = k_x^2 A \quad I_y = k_y^2 A \]

Composite Areas

Since the second moment is an integral, and since the integral over a sum of areas equals the sum of the integrals over the individual areas, we can find the second moment of a composite area by adding the integrals over the individual areas. Integrals over holes can be subtracted.
For the triangular area shown, find the following.

\[ I_x \]
\[ I_y \]
\[ k_x \]
\[ k_y \]
\[ J_O \]
\[ I_{Cx} \]
\[ I_{Cy} \]

\[ I_x = \int_A y^2 dA = \int_y y^2 (a-x) dy \]
\[ x = \frac{a}{b} y \]
\[ \therefore I_x = \int_0^b y^2 \left(a - \frac{a}{b} y\right) dy = a \left(\frac{b^3}{3} - \frac{b^4}{4b}\right) = \frac{ab^3}{12} = \frac{A b^2}{6} \]
\[ A = \frac{1}{2} ab \]

\[ I_y = \int_A x^2 dA = \int_x x^2 y dx \]
\[ y = \frac{b}{a} x \]
\[ I_y = \int_0^a x^2 \left(\frac{b}{a} x\right) dx = \frac{ba^3}{4} = A \frac{a^2}{2} \]

\[ k_x = \sqrt{\frac{I_x}{A}} = \frac{b}{\sqrt{6}} \]
\[ k_y = \sqrt{\frac{I_y}{A}} = \frac{a}{\sqrt{2}} \]
\[ J_O = I_x + I_y = A \left(\frac{b^2}{6} + \frac{a^2}{2}\right) = \frac{A}{2} \left(\frac{b^2}{3} + a^2\right) \]
For the T-shaped cross section shown, find the second moments about the \(x'\) and \(y'\) axes.

- Divide area into parts.
- Centroid is on the \(y\) axis.
- Compute \(y\) coordinate of centroid.
- Compute SAM of each part about an axis through its own centroid and parallel to the \(x\) axis.
- Transfer each SAM to the \(x'\) axis.
- Sum the results.

\[
I_{x} = I_{C_{x}} + A d_{y}^2 \Rightarrow I_{C_{x}} = I_{x} - A d_{y}^2
\]

\[
I_{C_{x}} = A \frac{b^5}{6} - A \left( \frac{b^3}{3} \right)^2 = A \frac{b^5 - 5b^3}{18} = \frac{ab^3}{36}
\]

\[
I_{y} = I_{C_{y}} + A d_{x}^2 \Rightarrow I_{C_{y}} = I_{y} - A d_{x}^2
\]

\[
I_{C_{y}} = A \frac{a^5}{2} - A \left( \frac{2a^3}{3} \right)^2 = A \frac{a^5 - 2a^3}{18} = \frac{a^3b}{36}
\]

\[
y = \frac{A_{1} y_{1} + A_{2} y_{2}}{A_{1} + A_{2}} = \frac{(bh) \left( \frac{h_{1} + h_{2}}{2} \right) + (bh) \left( \frac{h_{2}}{2} \right)}{300(50) + 250(50)} = \frac{2275}{11} = 206.8 \text{ mm}
\]

\[
I_{C_{x}} = (I_{C_{x}})_{1} + (I_{C_{x}})_{2}
\]

\[
(I_{C_{x}})_{1} = \left( \frac{bh^3}{12} \right)_{1} + A d_{y}^2 = \left( \frac{bh^3}{12} \right)_{1} + (bh) \left( \frac{y_{1} - y}{2} \right)^2
\]

\[
(I_{C_{x}})_{2} = \left( \frac{bh^3}{12} \right)_{2} + A d_{x}^2 = \left( \frac{bh^3}{12} \right)_{2} + (bh) \left( \frac{y_{2} - y}{2} \right)^2
\]

\[
I_{C_{x}} = 222 \times 10^4 \text{ mm}^4
\]

\[
I_{C_{y}} = (I_{C_{y}})_{1} + (I_{C_{y}})_{2}
\]

\[
(I_{C_{y}})_{1} = \left( \frac{bh^3}{12} \right)_{1} \quad \text{and} \quad (I_{C_{y}})_{2} = \left( \frac{bh^3}{12} \right)_{2}
\]

\[
I_{C_{y}} = 115 \times 10^4 \text{ mm}^4
\]
Second Areal Moment

2 We have seen in online homework that when computing the real MOI without having mass per unit area, we are essentially computing the equivalent of the SAM.

3 This fact was used in the previous notes on MOI.

4 These equations should look familiar to you. They are the same as they were for MOI, except that the differential element of mass in the integrals has been replaced by a differential element of area.

5 The SAM is used primarily in analyzing the strength of structural members. You will encounter this in a course that may have any of several different names, the most common of which are “mechanics of materials,” “strength of materials,” “solid mechanics” and “mechanics of deformable bodies.” In such courses, the focus is on the properties of a structural member’s cross section, which is a planar area.

6 When considering the twisting of a shaft by a torque, the polar second moment is the property of the cross section that matters.
7 This is the same method that we use to transfer the real MOI from an axis through the mass center to a parallel axis.

8 The transfer term is the same for the SAM as it was for the MOI, except that the square of the transfer distance is multiplied by the area rather than the mass.

9 This is true for both the MOI and the SAM.

10 This should look familiar. It is the same as it was for the MOI except that the mass had been replaced by the area.

11 Again, these are the same as they were for the MOI except that the mass has been replaced by the area.

12 This is the same principle that we used to compute the MOI for composite masses.

13 In the following slides, we will compute the first five quantities from their definitions. The last two will be computed using the transfer theorem.
14 For an element of area that is parallel to the $x$ axis, the differential in the integral becomes $dy$, rather than $dA$. So, the variable of integration becomes $y$, which ranges from zero to $b$. The value of $y$ is the same for the entire element. But each element has a different length, because the left end changes position as the value of $y$ changes during the integration. But the left end is on the slanted line, for which we can easily write an equation that relates $x$ and $y$. Substitution then gives an integrand that depends only on $y$ and constants. Then we need only the power law of integration to complete evaluation of the integral.

15 The approach used here is similar to that seen on the previous slide, except that the element of area is chosen parallel to the $y$ axis, so that the entire element is at the same value of $x$.

Comparing the results on this slide and the previous slide clearly shows that if the sides $a$ and $b$ were equal in length, we would have a larger SAM about the $y$ axis than about the $x$ axis. Inspection of the figure shows why this is the case. The axes have been chosen so that most of the area is farther from the $y$ axis than it is from the $x$ axis. That must lead to a larger SAM about the $y$ axis.

16 These results follow directly from definitions.

17 To compute the SAM about axes through the triangle’s centroid, we could repeat the integration process using different ranges for $x$ and $y$. For $y$, the range would be from $-b/3$ to $+2b/3$, and for $x$ the range would be from
But the transfer theorem gives a simpler approach because we already have SAM about the $x$ and $y$ axes and we know the distances needed to transfer those to parallel axes through the centroid.

Note that to transfer the SAM about the $x$ axis, we need a distance parallel to the $y$ axis, and vice versa.

While the two results shown here appear similar, they would be equal only if the lengths $a$ and $b$ were equal.

In each case, the cubed length is the one perpendicular to the axis about which the SAM is computed.

18 It is convenient to divide the area into two rectangles. We need the centroid of the overall cross-sectional area so that we can transfer the SAM of each rectangle from its centroid to the overall centroid.

Symmetry gives the $x$ coordinate of the centroid.

19 For each rectangle, $b$ denotes the side parallel to the $x$ axis and $h$ denotes the side parallel to the $y$ axis.

Once we have the overall centroid, we can compute the SAM of each rectangle about its own centroid and then transfer it to the overall centroid.

For the SAM about the $y$ axis, no transfer terms are needed. This is because the overall centroid and the centroids of both rectangles are on the $y$ axis. Note that $b$ and $h$ are still parallel to the $x$ and $y$ axes, respectively. In the SAM of a rectangle about an axis through its centroid, the cubed length is the one perpendicular to the given axis.