Moments - 2

Moment Arm

For any point \( Q \) on the line of action of \( \vec{F} \),
\[
d = |\vec{r}_{Q/P}| \sin \theta_Q
\]

Cross Product

For any two vectors,
\[
\vec{A} \times \vec{B} = (|\vec{A}| |\vec{B}| \sin \theta) \hat{u}
\]
determined by RHR

\[
\vec{r} \times \vec{F} = (|\vec{r}| |\vec{F}| \sin \theta) \hat{u}
\]

By inspection and from previous results,
\[
\vec{r}_{A/P} \times \vec{F} = (|\vec{r}_{A/P}| |\vec{F}| \sin \theta_A) \hat{u} = (|\vec{F}|d \hat{u} = \vec{M}_P
\]

Likewise,
\[
\vec{r}_{B/P} \times \vec{F} = \vec{M}_P
\]

So, for any point on the line of action of the force,
\[
\vec{M}_P = \vec{r} \times \vec{F}
\]

Note: \( \vec{r} \) begins (tail) at the moment center.
Principle of Moments

• The moment of a sum of forces equals the sum of the moments of the individual forces.

• Also called Varignon’s Theorem

\[ \vec{M}_p = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots = \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \cdots) \]

Couples

• A couple is a pair of equal and opposite forces.

• The net (resultant) force of a couple is zero.

• The net (resultant) moment of a couple about any moment center is the magnitude of one of the forces times the distance between the forces.

Couples

The net moment of this planar couple is the same about any moment center in the plane.

For planar problems, it is acceptable to denote a unit vector into or out of the plane with (CW) or (CCW), respectively, or with curved arrows denoting those directions.

\[ \vec{M} = (Fd)\hat{u} = Fd(CCW) = Fd \]

3D Moments

Curved arrows are ambiguous for representing the directions of three-dimensional moments. But double-headed arrows are unambiguous. The senses (directions) of the moments represented by the double-headed arrows in the figure are determined by the right hand rule.
Moment about a Line

Q: If the moment of a force about a point \( P \) is \( \mathbf{M}_P \), what is the moment of the force around a specified line passing through \( P \)?

A: It is the projection of the moment about \( P \) onto the direction of the line.

Each component is the projection of the moment (red) onto an axis. Each component is the moment around one of the axes.

From the previous slide and what we know about projecting a vector along a specified line, it follows that for a moment \( \mathbf{M}_P \), the part of that moment that is about a specified line through \( P \) is

\[
\left( \mathbf{M}_P \cdot \hat{u} \right) \hat{u}
\]

where \( \hat{u} \) is a unit vector pointing along the specified line.

To find the moment of a force about a line, we can use any point \( P \) on the line!
Moments - 2

2 To compute the moment of \( \vec{F} \) about the point \( P \), we need the length, \( d \), of the moment arm. In the right triangle \( PAC \), the side with length \( d \) is opposite the angle \( \theta_A \) (by virtue of equal angles at intersecting lines). The hypotenuse of that triangle is the magnitude of the vector \( \vec{r}_{A/P} \), which is the position of point \( A \) relative to point \( P \). These facts and the definition of the sine function lead directly to the first equation at the right of the figure.

Also, in the right triangle \( PBC \), the side with length \( d \) is opposite the angle \( \theta_B \), the hypotenuse is the magnitude of \( \vec{r}_{B/P} \) and this leads to the second equation at the right of the figure.

The process described above can be repeated for any point on the line of action of the force and will always result in an equation like the one at the bottom of the slide.

3 In previous notes on vector operations, we have learned that the product \( \vec{A} \times \vec{B} \) can be written as the magnitude of \( \vec{A} \) times the magnitude of \( \vec{B} \) times the sine of the angle between them times a unit vector determined by the right hand rule. For the product \( \vec{r} \times \vec{F} \), this gives the equation at the bottom of the slide. On the next slide, we will use this result in conjunction with slide 2.
Here we see that the moment of the force $\vec{F}$ about $P$ can be calculated as the cross product of a position vector with $\vec{F}$. The position vector must begin at the moment center, $P$, and may end at any point on the line of action of $\vec{F}$. Note that in the final, boxed equation, the moment center is identified by the subscript on the symbol for the moment. Except in one circumstance (to be discussed on slide 7), we must always include the subscript identifying the moment center. Any single force has different moments about different points. The moment calculation is meaningless unless we specify the point about which the moment is being calculated.

When we need to compute the net moment of a set of forces about some moment center, we have two choices. We could compute each force’s moment and then add them together. Or, we could first sum the forces to obtain the resultant, and then compute the moment of that. The result will be the same because the moment of the sum of the forces equals the sum of the moments of the individual forces.

From the definition of a couple given here, it should be clear that a couple exerts no net force. But it may not be obvious that the moment of a couple is the same no matter what point we choose as a moment center. The next slide should help to clarify this.

If we compute the net moment of the couple about some point on the line of action of the force on the left, we get $Fd \, (CCW)$. (The force on the left has no moment about the point in question.) If we compute the net
moment of the couple about some point on the line of action of the force on the right, we get $Fd$ (CCW). (The force on the right has no moment about the point in question.) If we compute the net moment of the couple about some point between the lines of action of the two forces on the left, we get $Fd$ (CCW). (Each force is multiplied by a moment arm that is part of the distance between the forces. The two moment arms add to equal the total distance between the forces.) It can be proven that this result is the same no matter what point is chosen as the moment center. So, for the moment of a couple, it is not necessary to specify a moment center.

In the final line of the equation, the direction of the moment is indicated with a curved, counterclockwise arrow. The directions of moments that are all perpendicular to a single plane can easily be indicated in this way. But that is not so for three-dimensional moments.

8 When the thumb of the right hand points in the direction of a double-headed arrow, the sense of the represented moment is in the direction that the fingers curl when being closed. For each of the moments in the figure, when viewed from the head looking towards point $P$, the moment’s direction is counterclockwise.

9 When we represent moment vectors with double-headed arrows, components are found just as they are for any other vectors.
The moment of a force about a specified line depends on the perpendicular distance from the specified line to the line of action of the force. That distance does not change by moving point $P$ along the specified line. That is why the formula given here produces the same result for any point on the specified line.