Adding 3-D Forces

- Using **Cartesian components**, we can easily add any number of **3-D (non-coplanar)** forces.
- This is the **most convenient** way to accomplish the task.

What are the magnitude and direction cosines of the resultant force?

\[ \vec{F}_1 + \vec{F}_2 = ? \]

\[ \vec{F}_1 = \vec{F}_{1\hat{u}} = \vec{F}_{1\hat{u}/A} \]
\[ = \left( 80 \text{ lb} \right) \frac{-2.5\hat{i} - 4\hat{j} + 6\hat{k}}{\sqrt{(-2.5)^2 + (-4)^2 + 6^2}} \]
\[ = \frac{80 \text{ lb}}{\sqrt{58.25}} (-2.5\hat{i} - 4\hat{j} + 6\hat{k}) \]

\[ \vec{F}_2 = \vec{F}_{2\hat{u}} = \vec{F}_{2\hat{u}/A} \]
\[ = \left( 50 \text{ lb} \right) \frac{-2\hat{i} - 4\hat{j} - 6\hat{k}}{\sqrt{2^2 + (-4)^2 + (-6)^2}} \]
\[ = \frac{50 \text{ lb}}{\sqrt{56}} (2\hat{i} - 4\hat{j} - 6\hat{k}) \]

\[ \vec{F}_1 + \vec{F}_2 = \left[ \begin{array}{c} \frac{-2.5(80)}{\sqrt{58.25}} + \frac{2(50)}{\sqrt{56}} \\ \frac{-4(80)}{\sqrt{58.25}} + \frac{-4(50)}{\sqrt{56}} \\ \frac{6(80)}{\sqrt{58.25}} + \frac{-6(50)}{\sqrt{56}} \end{array} \right] \text{ lb} \]
\[ = (-12.842\hat{i} - 68.654\hat{j} + 22.803\hat{k}) \text{ lb} \]

\[ |\vec{F}_1 + \vec{F}_2| = \sqrt{(-12.842)^2 + (-68.654)^2 + (22.803)^2} \text{ lb} = 73.5 \text{ lb} \]

\[ \vec{F}_1 + \vec{F}_2 = |\vec{F}_1 + \vec{F}_2| \left[ \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k} \right] \]

\[ \cos\alpha = -12.842 = -0.175 \quad \alpha = 100' \]
\[ \cos\beta = -68.654 = -0.934 \quad \beta = 159' \]
\[ \cos\gamma = 22.803 = 0.310 \quad \gamma = 71.9' \]
Force Projection

• Any force can be resolved into components parallel and perpendicular to any specified direction.

• The component of a force in a particular direction (also called the projection of the force onto the direction) is the product of:
  1. the magnitude of the force,
  2. the cosine of the angle between the force and the direction, and
  3. a unit vector in the specified direction.

Scalar Projection

• The scalar projection of a force onto a direction is the magnitude of the force times the cosine of the angle between the force and the specified direction, which is the same as the dot product of that force with a unit vector in that direction.

\[ \vec{F} \cdot \hat{u} = |\vec{F}| |\hat{u}| \cos \theta = F \cos \theta \]

Vector Projection

• As a vector, the projection of a force onto a specified direction is the scalar projection times a unit vector in the specified direction.

\[ (F \cos \theta) \hat{u} = (\vec{F} \cdot \hat{u}) \hat{u} \]

What is the vector projection of the force along the pole in the direction from \( O \) to the pole’s lower end?

\[ \hat{u} = \frac{2\hat{i} + 2\hat{j} - \hat{k}}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{1}{3} (2\hat{i} + 2\hat{j} - \hat{k}) \]

\[ [\vec{F} \cdot \hat{u}] = \left[ \left( 2\hat{i} + 4\hat{j} + 10\hat{k} \right) \text{kN} \right] \left[ \frac{1}{3} (2\hat{i} + 2\hat{j} - \hat{k}) \right] \hat{u} \]

\[ = \frac{2\cdot2 + 4\cdot2 + 10\cdot(-1)}{3} \text{kN} \hat{u} = \frac{2}{3} \text{kN} \hat{u} \]

\[ \hat{u} = \frac{2}{9} \text{kN} (2\hat{i} + 2\hat{j} - \hat{k}) = \frac{4}{9} \hat{i} + \frac{4}{9} \hat{j} - \frac{2}{9} \hat{k} \text{kN} \]

\[ |\vec{F}| = \sqrt{2^2 + 4^2 + 10^2} \text{ kN} \]

\[ = \sqrt{120} \text{ kN} = 10.954 \text{ kN} \]
Non-coplanar (3D) Forces

2 The process for adding non-coplanar forces is the same as that for adding coplanar forces, except that the vectors have three Cartesian components.

3 On this slide, the unit vectors pointing in the directions of $\vec{F}_1$ and $\vec{F}_2$ are denoted by $\hat{u}_1$ and $\hat{u}_2$. They are computed using techniques discussed in previous notes about vectors. You should verify these calculations. You should be able to see why each term in each equation is as shown here.

4 The resultant is calculated in the first two equations on this slide by adding the equations for the two forces found on the previous slide, collecting components and performing the indicated arithmetic operations. But the resultant is not what we were asked to find. We were asked to find the magnitude and direction cosines of the resultant. The magnitude is shown in the first red box, with the answer given to three significant figures. In this course, all answers should be correct to three significant figures. (That is not the same as being correct to within 1%. If, for example, an exact answer is 1.00, then any number between 0.99 and 1.01 is correct to within 1%. But only 1.00 is correct to three significant figures.) Note that to produce an answer correct to three signifi-
cant figures, it is necessary to carry several more significant figures in all arithmetic operations leading to the answer.

The third equation on this slide shows the resultant in the magnitude and unit vector form. You should see that the direction cosines can be obtained by factoring out (which means dividing by) the magnitude. The results shown in the second red box are correct to three significant figures, and were obtained to this accuracy by using five significant figures in the calculations.

The angles shown at the bottom right of the slide were not requested in this problem. But you should be able to compute them using the inverse cosine function. The angles $\alpha$, $\beta$ and $\gamma$ are, respectively, the angles that the resultant makes with the $x$, $y$ and $z$ axes. Each of the numbers shown is correct to three significant figures.

5 When we write a force, or any other vector, as the sum of parts pointing in different directions, the parts are called **vector components** and the process of finding them is referred to as **resolving** the force or vector into components. Using the information in the second bullet on this slide, we can find the component of any force in any direction. The vector component is a scalar multiplied by a unit vector pointing in the specified direction. The scalar is referred to as the **scalar projection** of the force onto the specified direction.
The scalar projection and can be positive, negative or zero. If the scalar projection is positive, the component points in the specified direction. If negative, the direction of the component is opposite to the specified direction. If the scalar projection is zero, none of the force is parallel to the specified direction. (This implies that the force is in a plane perpendicular to the specified direction.)

Note that when the angle between the specified direction and the force is less than, equal to or greater than 90°, the scalar projection is positive, zero or negative, respectively.

The vector component is the part of the force that is parallel to the specified direction. It is also sometimes called the vector projection. It is simply the scalar projection multiplied by a unit vector in the specified direction.
8 We can solve this problem using the formula given on the previous slide. First, we need \( \hat{u} \), the unit vector pointing from point \( O \) along the pole. Then, we use the formula on the previous slide and compute the needed dot product using what we know about the dot products of the unit vectors along the coordinate axes. The requested answer is given in the two red boxes.

Note that in the next to last line on the left, the scalar projection of the force is found to be \( \frac{2}{3} \text{kN} \). (Since it is positive, the vector projection points in the direction of \( \hat{u} \).) Note also that the magnitude of the entire force, shown at bottom right, is much greater than the scalar projection along the pole. This implies that almost all of the force is perpendicular to the pole.