DEFINITIONS

Year-end convention: Expenses occurring during the year are assumed to occur at the end of the year.

CASH FLOW DIAGRAMS

Some FE exam problems may use cash flow diagrams. They can also be useful to represent a word problem graphically. The flow of cash is shown as arrows on a time line scaled to the magnitude of the cash flow. Expenses are down arrows, and receipts are up arrows.

Example: Cash flow diagram

A mechanical device will cost $20,000 when purchased. Maintenance will cost $1000 per year. The device will generate revenues of $5000 per year for 5 years. The salvage value is $7000. Draw and simplify the cash flow diagram.

PRESENT WORTH

The value (at the current time) of some value to be realized in the future.

Example: Present worth

How much is that $11,000 at the end of five years worth to you today, given a 10% effective annual interest rate?

Using the formula in the factor conversion table:

\[ P = F(1 + i)^n = (11,000)(1 + 0.1)^5 = 6831 \]

or using the factor table for 10%:

\[ P = (P/F,10\%,5) = (11,000)(0.6209) = 6831 \]

How much is that series of $4,000 at the end of the first four years worth to you today, given a 10% effective annual interest rate?

Using the formula in the factor conversion table:

\[ P = A[(1 + i)^n - 1]/[i(1+i)^n] = (4,000)(0.4641/0.14641) = 12,680 \]

or using the factor table for 10%:

\[ P = (P/A,10\%,4) = (4,000)(3.1699) = 12,680 \]

Thus, the Present Worth of the entire cash flow is as follows:

\[ PW = P_0 + P/F + P/A = -20,000 + 6,831 + 12,680 = -489 \]
COMPOUNDING

Using equivalence equations

If there is a factor table for the interest rate in the NCEES FE Reference Handbook, use it; don’t waste time using the equivalence equations. But sometimes the problem will have an interest rate for which there is no table. While it is possible to interpolate between the tables—for example, if the interest rate is 5% you can use the average between the 4% table and the 6% table—errors occur because the relations are not linear functions of interest rate.

Nonannual compounding

An interest rate that is compounded more than once in a year is converted from the compound nominal rate to an annual effective rate.

\[
 ie = \left(1 + \frac{r}{m}\right)^m - 1
\]

\[
 ie = \text{effective annual rate}
\]
\[
 r = \text{nominal rate per period}
\]
\[
 m = \text{number of compounding periods in a year}
\]

After the interest rate is converted to an annual rate, the problem is solved as any annual interest rate problem using the factor conversion formulas or the factor tables. This is true if the cash flows in question are annual. If they are in terms of the compounding period, use the period rate, which is \( \frac{r}{m} \).

The interest rate must be in decimal (not percentage) form.

Example: Nonannual compounding

A savings and loan offers a 5.25% rate per year compounded daily over 365 days per year. What is the effective annual rate?

\[
 ie = \left(1 + \frac{r}{m}\right)^m - 1 = \left(1 + \frac{0.0525}{365}\right)^{365} - 1 = 0.0539
\]

Discount factors for continuous compounding

The formulas for continuous compounding are the same formulas in the factor conversion table with the limit taken as the number of periods, \( n \), goes to infinity. Solve continuous compounding problems with formulas in the NCEES FE Handbook.

COMPARISON OF ALTERNATIVES

Present worth

When alternatives do the same job and have the same lifetimes, compare them by converting each to its cash value today. The superior alternative will have the highest present worth.

Example: Present worth

Investment A costs $10,000 today and pays back $11,500 2 years from now. Investment B costs $10,000 today and pays back $5500 each year for 2 years. If the interest rate of 6% is used for comparison, which investment is superior?
\[ P(A) = -10,000 + (11,500)(P/F, 6\%, 2) \]
\[ = -10,000 + (11,500)(0.89) \]
\[ = -10,000 + 10,235 = 235 \]

\[ P(B) = -10,000 + (5,500)(P/A, 6\%, 2) \]
\[ = -10,000 + (5,500)(1.8334) \]
\[ = -10,000 + 10,083 = 83 \]

\[ 235 > 83, \text{ so Alternative A is superior.} \]

**Capitalized costs**

Used for a project with infinite life that has repeating expenses every year.

Compare alternatives by calculating the capitalized costs (i.e., the amount of money needed to pay the start-up cost and to yield enough interest to pay the annual cost without touching the principle.)

NOTE: The factor conversion for a project with no end is the limit of the F/A factor as the number of periods, \( n \), goes to infinity:

**Example: Capitalized cost**

What is the present worth of a public works project that will cost $25,000,000 now and will require $2,000,000 in maintenance annually? The effective interest rate is 12%.

**Annual cost**

Used for alternatives that do the same job but have different lives. Compare the cost per year of each alternative.

The alternatives are assumed to be replaced at the end of their lives by identical alternatives.

**Example: Annual cost**

Brick will last 30 years, cost $1,800, and require $5/yr maintenance. Wood will last 10 years, cost $450, and require $20/yr maintenance. Which is superior over 30 years? The interest rate is 8%.

\[ \text{EUAC (brick)} = (1800)(A/P, 8\%, 30) + 5 \]
\[ = (1800)(0.0888) + 5 = 165 \]

\[ \text{EUAC (wood)} = (450)(A/P, 8\%, 10) + 20 \]
\[ = (450)(0.149) + 20 = 87.50 \]

(EUAC = Equivalent Uniform Annual Cost)

\[ 87.50 < 165, \text{ so Wood is superior.} \]
ALTERNATIVES COMPARED TO A STANDARD

Cost (C) – benefit (B) analysis

\[ B - C > 0 \text{ or } B/C > 1 \]

Example: Cost-benefit analysis

The initial cost of a proposed project is $40M, the capitalized perpetual annual cost is $12M, the capitalized benefit is $49M, and the residual value is $0. Should the project be undertaken?

\[
B = \$49M, \ C = \$40M + \$12M + \$0 \\
B - C = \$49M - \$52M = -\$3M < 0
\]

The project should not be undertaken.

Rate-of-return problems

The rate of return on an investment must exceed the minimum attractive rate of return (MARR). The rate of return is calculated by finding an interest rate that makes the present worth zero. Often this must be done by trial and error.

\[ PW \ (i\%) = 0; \text{ if } i\% \geq MARR, \text{ accept.} \]

Example: Rate of Return

\[ PW = -\$100 + \$3(P/A, i\% ,3) + \$130(P/F, i\% ,3) = 0 \]
\[ $33.33 = (P/A, i\% ,3) + 43.33(P/F, i\% ,3) \]

\[ i\% = 8\% \]

\[ (P/A, 8\% ,3) + 43.33(P/F, 8\% ,3) = 2.5771 + 43.33(0.7938) = 37.24 > 33.33 \]

\[ i\% = 10\% \]

\[ (P/A, 10\% ,3) + 43.33(P/F, 10\% ,3) = 2.4869 + 43.33(0.7513) = 32.50 < 33.33 \]

\[ i - 0.10/33.33 - 32.50 = 0.10/0.08/32.50 - 37.24 \]

\[ i\% = 46.234/47.4 = 0.975 \]

BREAK-EVEN ANALYSIS

Calculating when revenue is equal to cost, or when one alternative is equal to another if both depend on some variable.

Example: Break-even analysis

How many kilometers must be driven per year for leasing and buying to cost the same? Use 10% interest and year-end cost. Leasing: $0.15 per kilometer Buying: $5000 purchase cost, 3-year life, salvage $1200, $0.04 per kilometer for gas and oil, $500 per yr for insurance.
EUAC (leasing) = $0.15x where x is kilometers driven
EUAC (buying) = $0.04x + $500 + ($5 k)(A/P,10\%,3) − ($1.2 k)(A/F,10\%,3) = $0.04x + $2148
Setting EUAC (leasing) = EUAC (buying) and solving for x
$0.15x = $0.04x + $2148
x = 19,527 km that must be driven to break even

INFLATION
Making a correction to the interest rate for inflation.
\[ d = i + f + (i \times f) \]

To determine the value of P dollars n years from today, use the following: \( F = P(1+f)^n \)
If interest is being compounded during the inflationary period, \( F = P \left[\frac{(1+i)}{(1+f)}\right]^n \)

Example: Inflation
1. An item presently costs $100. If the inflation rate is 6%, how much will it cost in 5 years?
Solution \( F = $100 \times (1.06)^5 = $133.82. \)

DEPRECIATION

Straight line
The depreciation per year is the cost minus the salvage value divided by the years of life.
\[ D_j = \frac{C - S_j}{n} \]

Accelerated Cost Recovery System (ACRS)
The depreciation per year is the cost times the ACRS factor (see the table in the NCEES Handbook). Salvage value is not considered.
\( D_j = (ACRS \text{ factor}) \times C \)

Example: Straight-line and ACRS
An asset is purchased that costs $9000. It has a 10-year life and a salvage value of $200. Find the straight-line depreciation and ACRS depreciation for 3 years.

- Straight-line depreciation /year = ($9000 – $200)/10 = $880/yr (each year)
- ACRS depreciation
  First year ($9000)(0.1) = $ 900
  Second year ($9000)(0.18) = $1620
  Third year ($9000)(0.144) = $1296

Book value
Assumed value of the asset after j years. The initial cost minus the sum of the depreciations out to the j\textsuperscript{th} year
\[ BV = \text{initial cost} - \sum D_j \]

Example: Book value

What is the value of the asset in the previous example after 3 years using straight-line depreciation? Using ACRS depreciation?

- Straight line
  \[ $9000 - (3)($800) = $6360 \]

- ACRS
  \[ $9000 - $900 - $1620 - $1296 = $5184 \]

TAX CONSIDERATIONS

Expenses and depreciation are deductible, revenues are taxed. Example: Tax considerations. A corporation pays 53% income tax on profits. It invests $10,000 in an asset that will provide $3000 in revenue per year for 8 years. The annual expenses are $700, the salvage value is $500, and 9% interest is used. What is the after-tax present worth? Disregard depreciation.

\[
\begin{align*}
P &= -$10,000 + ($3000)(PA, 9\%, 8)(1 - 0.53) - ($700)(PA, 9\%, 8)(1 - 0.53) + ($500)(PF, 9\%, 8) \\
&= -$10,000 + ($3000)(5.5348)(0.47) - ($700)(5.5348)(0.47) + ($500)(0.5019) = -$3766
\end{align*}
\]

Tax credit

A one-time benefit from a purchase that is subtracted from income taxes.

Example: Tax credit

An investment costs $5000 and is not depreciable, but there is a one-time 20% tax credit. In the same year, revenue is $45,000, and expenses (excluding the $5000 investment) are $25,000. The income tax rate is 53%, and the interest rate is 10%. What is the after-tax present worth for the year?

Since 20% of $5000 is $1000, the corporation will get $1000 credit against its taxable income.

\[
\begin{align*}
P &= -$5k + (($45k - $25k)(1 - 0.53) + ($1k)(PF, 10\%, 1) \\
&= -$5k + ($10.4k)(0.9091) = $4455
\end{align*}
\]

BONDS

Bond value = present worth of payments over the life of the bond.

Bond yield = equivalent interest rate of the bond compared to bond cost.

Example: Bonds

What is the maximum an investor should pay for a 25-year bond with a $20,000 face value and 8% coupon rate (with interest paid semiannually)? The bond will be kept to maturity. The effective annual rate for comparison is 10%.

\[
\begin{align*}
\phi_{\text{real}} &= r\%/\text{periods} = 8\%/2 = 4\% \text{ (effective rate per period)} \\
\text{Coupon payment} &= ($20,000)(\phi_{\text{bond}}) = $800 \\
The \text{comparison interest rate} &= 0.10 = (1 + \phi)^\frac{1}{2} - 1 \\
\text{Solving for} \phi &= (0.10 + 1) - 1 = 0.0488 \\
\text{There are 50 periods, so}\ P &= ($800)(PA, 4.88\%, 50) + ($20,000)(PF, 4.88\%, 50) \\
P &= ($800)(18.600) + ($20,000)(0.09233) = $16,727
\end{align*}
\]

The investor should pay no more than $16,727 for the bond.
(1) What is the uninflated present worth of $2000 in 2 years if the average inflation rate is 6% and i is 10%?

Answer: $1471

(2) It costs $75 per year to maintain a cemetery plot. If the interest rate is 6.0%, how much must be set aside to pay for maintenance on each plot without touching principle?

Answer: $1250

(3) It costs $1,000 for hand tools and $1.50 labor per unit to manufacture a product. Another alternative is to manufacture the product by an automated process that costs $15,000, with a $0.50 per-unit cost. With an annual production rate of 5,000 units, how long will it take to reach the break-even point?

Answer: 2.8 years

(4) A loan of $10,000 is made today at an interest rate of 15%, and the first payment of $3,000 is made 4 years later. The amount that is still due on the loan after the first payment is most nearly:

Answer: $14,490

(5) A machine is purchased for $1,000 and has a useful life of 12 years. At the end of 12 years, the salvage value is $130. By straight-line depreciation, what is the book value of the machine at the end of 8 years?

Answer: $420

(6) The maintenance cost for an investment is $2,000 per year for the first 10 years and $1,000 per year thereafter. The investment has infinite life. With a 10% interest rate, the present worth of the annual disbursement is most nearly:

Answer: $1480