Objectives
To illustrate the principle of linear impulse/linear momentum
To illustrate the principle of conservation of linear momentum and contrast it with the principle of conservation of mechanical energy

Summary
We started dynamics of particles with a study of the kinematics of particle motion. Once we learned basic definitions and equations to describe the motion, we introduced Newton's Second Law of Motion

\[ \sum F = ma \]

to relate the resultant force acting on a body to the instantaneous acceleration of the center of mass of the body. This approach to problem solving allows us to calculate a specific force or acceleration (both are vectors), which is only good for a particular instant in time for which the FBD was drawn. Of course if forces on the FBD remain unchanged as the body moves, then the resultant force and acceleration remain constant over the path. If the resultant force changes with time or position (a body cannot change its position without a change in time), then the acceleration changes also.

We next disguised the second law in the form of the work-kinetic energy principle and then modified that relation further to obtain the work-energy principle in the form:

\[ T_1 + V_{g1} + V_{e1} + U'_{1-2} = T_2 + V_{g2} + V_{e2} \]

where \( U'_{1-2} = \int_{s_i}^{s_f} \sum F \, ds \) is the work of all forces acting on the body except the gravitational force (weight) and the force from an elastic spring. The gravitational potential energy and the elastic potential energy of a linear spring are given by, respectively, \( V_g = mgh \), and \( V_e = (1/2)kx^2 \) where \( h \) is positive upward from a specified datum and \( x \) is the amount of stretch or compression in the spring. The work-energy approach is convenient to use when we do not need to calculate accelerations, but rather when changes in position and speed are of interest. It is particularly useful when one or more forces change their magnitude as the body moves along its path of interest (such as the force from a spring attached to the body). Finally, we saw that if \( U'_{1,2} = 0 \), then the total mechanical energy (kinetic plus potential) at position 2 was equal to that at the beginning of the path, position 1, and thus mechanical energy was conserved over the path of interest. We note that work and energy are scalars as they do not possess a direction, but either or both of \( U'_{1,2} \) and \( V_g \) could be negative.

A third way of applying Newton's Second Law to a body of constant mass \( m \) is to write it in the form \( \sum F = ma = d/dt (mv) \) which leads to

\[ \int_{t_1}^{t_2} \sum F \, dt = \int_{mv_1}^{mv_2} d(mv) = mv_2 - mv_1 = G_2 - G_1 \]

where \( G = mv \) is defined as the linear momentum of the body and \( v \) is the velocity of the center of mass of the body. The integral \( \int_{t_1}^{t_2} \sum F \, dt \) is defined as the linear impulse acting on the body over the time period \( \Delta t = (t_2 - t_1) \) and represents force times time. Thus, a linear impulse acting on a body causes a change in the linear momentum of the body. Note this is a vector equation. We will examine it in several ways in this week's laboratory.
Procedure and Analysis

Some Comments Related to our Experimental Apparatus

An air track, sketched below, is a device on which "cars" can travel and ride on a cushion of air. Thus there is relatively little friction opposing the motion of the cars if the speeds are relatively low. Some of our cars will have "bumpers" made of spring stock attached to either or both ends in order to provide for an "elastic" impact with another car or with the end of the air track. Of course no impact is perfectly elastic in that there is always some dissipation of mechanical energy. However, you will find that for small deformations of the springs, there is little loss of mechanical energy (kinetic energy in our case) over the impact period.

You should examine the air track to see how it operates. It is precisely made to rigid specifications, can be leveled both longitudinally and laterally (why?), and should be treated like a precision (and very expensive!) piece of equipment. In particular, do not allow the cars or anything else to scratch the track nor to scratch each other and keep the impacts "mild" with relatively small deformations of the springs. Put the cars on the track gently and only when the air blower is on. DO NOT press down on the cars when they are on the track. There are three sizes of cars having masses of approximately 575g, 380g, and 185g. However, you should measure the mass of each car used as they vary somewhat depending on the sizes of the attached springs.

Task 1. Determine the Linear Impulse Acting on a Body. Estimate the Time-Average Force Acting on the Body.

a) Hold one of the cars with an attached spring against the spring at the end of the track, compressing the springs enough so that when the car is released it obtains a speed \( v_2 \) of approximately 0.5 to 1m/s.

b) Record the time for the car to travel a known distance along the track. Calculate its velocity, \( v_2 \), along the track assuming it is constant along the measured length. Is this a reasonable assumption? It would be if the air track were perfectly level and frictionless.

c) Determine the linear momentum of the car both before it was released, \( G_1 \), and as it traveled along the track, \( G_2 \). Remember linear momentum is a vector and should have units here of kg\(\cdot\)m/s or slug\(\cdot\)ft/sec depending on your system of units. \( G_1 = m v_1 = ( )() = \underline{\hspace{1cm}} \) \( G_2 = m v_2 = ( )() = \underline{\hspace{1cm}} \)

d) Since the linear impulse on the car must equal its change in linear momentum, you can now calculate the linear impulse on the car over the time period \( \Delta t = (t_2 - t_1) \) that the car had a change in its linear momentum. From a FBD of the car while it is in contact with the spring, you'll see that, assuming the frictional force from the air track to be negligible, the linear impulse came entirely from the force of the spring on the car. The SI units for linear impulse are N\(\cdot\)s. Are these units equivalent to those of \(\Delta G\), (kg\(\cdot\)m/s)?

Since \( \int_{t_1}^{t_2} F_x \, dt = m( v_2 - v_1) \), then \( \int_{t_1}^{t_2} F_x \, dt = \int_{t_1}^{t_2} F_y \, dt = m( v_{x2} - v_{x1}) \)

Note that: \( \int_{t_1}^{t_2} F_y \, dt = \int_{t_1}^{t_2} F_z \, dt = 0 \)

Thus, as a vector, \( \int_{t_1}^{t_2} F \, dt = m( v_{x2} - v_{x1}) = ( )() = \underline{\hspace{1cm}} \)
e) Estimate the time increment $\Delta t = (t_2 - t_1)$ you feel it took for the car to obtain its velocity $v_2$. Use this value, which is likely less than 0.1 second, to calculate the average force exerted on the car by the spring while it was being accelerated from rest to its velocity $v_2$; i.e.

$$
\mathbf{F}_{sp} \int_{t_1}^{t_2} \mathbf{F}_{sp} \, dt = \mathbf{F}_{sp} \Delta t = (\mathbf{F}_{sp})_{avg} \Delta t = (\mathbf{F}_{sp})_{avg} (t_2 - t_1)
$$

so $(\mathbf{F}_{sp})_{avg} = \frac{(\mathbf{F}_{sp})_{avg} (t_2 - t_1)}{\Delta t} = \ldots$

We call this a time-average force since we are dividing the linear impulse by the time period. If you are really observant, you'll see we also could have obtained this resultant time-average force on the car by multiplying its mass by its average acceleration (calculated by $\Delta v/\Delta t$).

f) Using your measured value of $v_2$, calculate the work done on the body by the springs as the body was accelerated from rest to velocity $v_2$. Hint: Even though you know nothing about the springs (they are likely nonlinear) you can calculate their combined work indirectly by using the principle of work/energy. Note: work is a scalar.

$$
U'_{1-2} = \int_{s_1}^{s_2} F_{sp} \, ds = \frac{1}{2} \mathbf{v}_2^2 - \frac{1}{2} \mathbf{v}_1^2 = \frac{1}{2} ( \ldots ) - \frac{1}{2} ( \ldots ) = \ldots
$$

Development of the Equation for Conservation of Linear Momentum for Two Colliding Bodies

Two bodies A and B which collide will either have a common velocity after the impact period (they move as one body), or they will have different velocities at the end of the impact period and move apart. Let the $x$-direction be along the line of impact (the line drawn between their mass centers during the impact period). Also, let the only significant forces in the $x$-direction acting on the bodies during impact be the equal and opposite forces each exerts on the other.

We will prove below that for the situation described above, linear momentum is conserved in the $x$-direction during the impact period. We start with a FBD of each body during the impact period and apply the x-component of the linear impulse/momentum equation to each body.

On A:

$$
\mathbf{F}_{sp} \int_{t_1}^{t_2} \mathbf{F}_{sp} \, dt = \mathbf{F}_{sp} \Delta t = \mathbf{m}_A (v_{Ax_2} - v_{Ax_1})
$$

On B:

$$
\mathbf{F}_{sp} \int_{t_1}^{t_2} \mathbf{F}_{sp} \, dt = \mathbf{F}_{sp} \Delta t = \mathbf{m}_B (v_{Bx_2} - v_{Bx_1})
$$

Add the two above equations to get:

$$
\int_{t_1}^{t_2} - \mathbf{F}_{sp} \, dt + \int_{t_1}^{t_2} \mathbf{F}_{sp} \, dt = 0 = \mathbf{m}_A (v_{Ax_2} - v_{Ax_1}) + \mathbf{m}_B (v_{Bx_2} - v_{Bx_1})
$$

or: $\mathbf{m}_A v_{Ax_2} + \mathbf{m}_B v_{Bx_2} = \mathbf{m}_A v_{Ax_1} + \mathbf{m}_B v_{Bx_1}$

i.e. $\sum G_{x_2} = \sum G_{x_1}$

The FBDs and equations above show us the linear impulse in the $x$-direction on car A is equal and opposite to the linear impulse on car B, so the sum of the two impulses is zero. Thus the change in linear momentum for A must be just the negative of the change for B and thus the total change for the system must be zero. We can say that linear momentum is conserved in the $x$-direction for the system of two bodies. It is very important to realize that in the equations developed above, $v_{Ax_1}$, $v_{Bx_1}$, $v_{Ax_2}$, and $v_{Bx_2}$ are considered to be positive in the positive $x$-direction.
Task 2. Demonstrate Conservation of Linear Momentum for a Plastic Collision Between Two Bodies

If two bodies collide and stick together, they will move as one body after the impact period. We call this plastic impact; it is usually the first type of collision studied by dynamics students. We hope it has an impact on you.

a) Apply a piece of velcro to the vertical ends (no bumpers) of two cars having equal mass. Place one car, B, near the center of the track and place the other car, A, near one end and give it a slight push toward the car at rest. The cars should be oriented so the taped ends will come in contact and stick to each other. Measure their velocities before and after collision. Let \( \hat{i} \) be in the direction A moves.

\[
\begin{align*}
\mathbf{v}_{B1} &= 0 \\
\mathbf{v}_{A1} &= \left( \frac{\Delta x}{\Delta t} \right) \hat{i} = \\
\mathbf{v}_{A2} &= \mathbf{v}_{B2} = \left( \frac{\Delta x}{\Delta t} \right) \hat{i} = 
\end{align*}
\]

b) Draw FBDs of each car during the impact period and note that the sum of the impulses along the line of impact (the x-direction) equals zero, and thus linear momentum is conserved in the x-direction for this system of two bodies. Write the equation for conservation of linear momentum along the track, the x-direction.

Top View:

\[
\begin{align*}
A & \quad B \\
\text{+ x} \quad \hat{i}
\end{align*}
\]

c) Assume your experimental value of \( \mathbf{v}_{A1} \) is correct and use your expression above to calculate \( \mathbf{v}_2 \). Note that \( \mathbf{v}_{A2} = \mathbf{v}_{B2} = \mathbf{v}_2 \). Compare to your measured value of \( \mathbf{v}_2 \).

d) Calculate below the kinetic energy \( T \) for the combined body A and B and compare it to the total kinetic energy of the system before impact. Where did some of the kinetic energy go? This leads us to a very important conclusion. Conservation of linear momentum does not imply that mechanical energy is conserved. In fact, mechanical energy is almost never conserved during the impact of bodies, as the total kinetic energy after impact is almost always less than before impact. Some kinetic energy is dissipated or lost due to the impact.

\[
\begin{align*}
\sum T_2 &= \frac{1}{2} (m_A + m_B) v_2^2 = \frac{1}{2} \left( \mathbf{v}_{A1}^2 + \mathbf{v}_{B1}^2 \right) = \frac{1}{2} (\mathbf{v}_{A1}^2 + \mathbf{v}_{B1}^2) + 0 = \\
\sum T_1 &= \frac{1}{2} m_A v_{A1}^2 + \frac{1}{2} m_B v_{B1}^2 = \frac{1}{2} \left( \mathbf{v}_{A1}^2 + \mathbf{v}_{B1}^2 \right) = 
\end{align*}
\]

e) If time permits, you can repeat this exercise for cars of different mass. You can also investigate the situation of both cars having a velocity before impact in the same or in opposite directions, but you will need to estimate both of their velocities before impact to calculate their common velocity after impact. Keep the speeds low -- under 1 m/s!

Task 3. Demonstrate an Elastic Collision Between Two Bodies

An elastic collision (more often called an elastic impact) between two bodies is one in which the sum of their mechanical energy before impact is equal to the sum after the impact period. That is, the total mechanical energy (kinetic energy in our case since there is no change in potential energy) for the system of two bodies is conserved over the impact period; no energy is dissipated during the collision. Thus an elastic collision is an "idealization", but one which can be closely approximated for some materials (coefficient of restitution of 1).

a) For this part, use two cars A and B of the same mass, each having bumpers oriented toward each other when they are placed on the track. Place one car, B, in the center of the track and keep it at rest. Place the other car, A, having equal mass at one end of the track and give it a slight shove toward the other end. The bumpers should be oriented so that they meet upon collision. Determine the velocities of the cars before and after impact. Let the +x direction be the direction A initially moves.

\[
\begin{align*}
\mathbf{v}_{A1} &= \Delta x \hat{i} = \left( \frac{\Delta x}{\Delta t} \right) \hat{i} = \\
\mathbf{v}_{B1} &= 0 \\
\mathbf{v}_{A2} &= \Delta x \hat{i} = \left( \frac{\Delta x}{\Delta t} \hat{i} = \\
\mathbf{v}_{B2} &= \Delta x \hat{i} = \left( \frac{\Delta x}{\Delta t} \hat{i} = 
\end{align*}
\]

b) Draw a FBD below of each car during the impact period. Verify the impulses are equal in magnitude but in opposite x-directions and thus linear momentum is conserved in the x-direction for the system of two bodies. We can now write the equation for conservation of linear momentum and call it Eq. (1). Since the bodies can only travel in the x-direction, the subscript “x” will be left off of the velocity terms.

\[ m_A v_{A1} + m_B v_{B1} = m_A v_{A2} + m_B v_{B2} \]

Since \( v_{B1} = 0 \),

\[ v_{A1} = v_{A2} + \frac{m_B}{m_A} v_{B2} \]  

Since we have assumed an “elastic” collision, mechanical energy is also conserved. Thus

\[ \Sigma T_1 + \Sigma V_{g1} + \Sigma V_{e1} = \Sigma T_2 + \Sigma V_{g2} + \Sigma V_{e2} \]

where \( \Sigma \) represents the sum for both bodies A and B. Since \( \Sigma V_{g1} = \Sigma V_{g2} \) and \( \Sigma V_{e1} = \Sigma V_{e2} = 0 \), our equation reduces to \( \Sigma T_1 = \Sigma T_2 \). Thus

\[ m_A v_{A1}^2 + m_B v_{B1}^2 = m_A v_{A2}^2 + m_B v_{B2}^2 \]

If we solve for \( v_{A1} \) from Eq. (1) and substitute this expression into Eqn (2), we obtain after simplifying

\[ v_{A2} = \frac{v_{B2} \left( 1 - \frac{m_B}{m_A} \right)}{2} \]  

For our case, \( m_A = m_B \) and thus from Eq.(3), \( v_{A2} = 0 \), and by Eq.(1) \( v_{B2} = v_{A1} \), meaning, of course, that \( v_{B2} = v_{A1} \). How does this compare with what you observed?

c) If time permits you can repeat part (a) with two cars of different masses (the largest and smallest cars will give the most dramatic results). Let \( v_{B1} = 0 \) and experimentally determine the velocities \( v_{A1}, v_{A2}, \) and \( v_{B2} \). Your experimental results should agree closely with the values predicted from Eqs. (1) and (3).

**Caution:** Remember, the total kinetic energy for the system is conserved only for elastic impacts. Thus Eqs. (2) and (3) above only hold when we have direct central elastic impact of two bodies where \( v_{B1} = 0 \).

**Task 4. Determine if an Impact is Elastic, Plastic, or Something In Between**

a) Drop a super ball to the floor and compare its height of rebound to that height from which you dropped the ball. The heights should be almost the same if the floor is rigid and we have a good super ball. What does this say about the speed of the ball after impact compared to its speed just before striking the floor? (Think of work-energy principles). Compare the total kinetic energy of the ball and floor just before impact to that just after impact. Was mechanical energy conserved, or nearly conserved during the impact period? If your answer was "yes" then you had an elastic impact.

b) Repeat step (a) for a tennis ball. Is the impact elastic, plastic or something in between?

c) Repeat step (a) for a glob (a good technical term!) of play dough, or silly putty if you have some, and observe the type of impact.

**Task 5. Observe Direct Central and Oblique Central Impact on the Air Hockey Table**

This is your last “task” in Physical Homework in EF 102! Enjoy -- and test your skill (or luck??). Everyone needs to participate, so the lab instructor at the air hockey table will limit your time until all have had a chance to produce direct central impact and oblique central impact with the hockey puck. Or should I say a chance to
“play”? Whatever, demonstrate the following (all sketches are top views of the pucks “A” and “B” and the strikers “S”):

**a) Direct central impact of two pucks.**
Let A strike B, which is initially at rest. Why does A stop and B take off with approximately the same velocity that A had? Is the coefficient of restitution between A and B approximately equal to 1?

![Sketch of direct central impact](image)

**b) Oblique central impact between the striker and the puck (two people can be doing this at the same time since we have two pucks and two strikers).**
Move the striker such that \(v_{S1}\) makes an angle of between 30 and 60 with the line of impact with the puck. Note the puck, which is initially at rest, obtains a velocity along the line of impact between the striker and the puck. Why? Two people can do this at the same time as indicated on the sketch.

![Sketch of oblique central impact](image)

**c) Oblique central impact with a wall.**
Strike the puck as shown. Observe the angles \(\alpha\) and \(\beta\). Why are they approximately the same? Try to score a goal in this manner. Two people can be doing this at the same time (each gets half a hockey table and one goal!).

![Sketch of oblique central impact with a wall](image)

You can go now (after convincing your lab instructors you know everything), and good luck on your exams!